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Comment on "Calculation of Quarkonium Spectrum and m_b, m_c to Order α_s^4"*

A. Pineda

Theory Division
CERN,
1211 Geneva 23, Switzerland

and

F. J. Ynduráin

Departamento de Física Teórica, C-XI,
Universidad Autónoma de Madrid,
Canto Blanco,
E-28049, Madrid, Spain.

Abstract

In a recent paper, we included two loop, relativistic one loop and second order relativistic tree level corrections, plus leading nonperturbative contributions, to obtain a calculation of the lower states in the heavy quarkonium spectrum correct up to, and including, $O(\alpha_s^4)$ and leading Λ^4/m^4 terms. The results were obtained with, in particular, the value of the two loop static coefficient due to Peter; this been recently challenged by Schröder. In our previous paper we used Peter's result; in the present one we now give results with Schröder's, as this is likely to be the correct one. The variation is slight as the value of b_1 is only one among the various $O(\alpha_s^4)$ contributions. With Schröder's expression we now have,

$$m_b = 5\,001^{+104}_{-66}\ \text{MeV}; \quad \bar{m}_b(\bar{m}_b^2) = 4\,440^{+43}_{-28}\ \text{MeV},$$

$$m_c = 1\,866^{+190}_{-154}\ \text{MeV}; \quad \bar{m}_c(\bar{m}_c^2) = 1\,531^{+132}_{-127}\ \text{MeV}\,.$$

Moreover,

$$\Gamma(\Upsilon \to e^+e^-) = 1.07 \pm 0.28\ \text{keV}\ (\text{exp.} = 1.320 \pm 0.04\ \text{keV})$$

and the hyperfine splitting is predicted to be

$$M(\Upsilon) - M(\eta) = 47^{+15}_{-13}\ \text{MeV}\,.$$

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1 Introduction

In a recent paper[1], to be hereafter denoted by PYI, we evaluated the heavy quarkonium spectrum to order α_s^4. The ingredients for the calculation were the one loop corrections[2] to the static potential, and its first and second order contributions to the spectrum; the relativistic, and mixed relativistic–one loop[3] corrections; the two loop corrections to the static potential, and the leading nonperturbative corrections[4]. The two loop correction to the static potential used was that calculated by Peter[5]; but the calculation of Peter has recently been challenged by Schröder[6]. In this last reference, the result of Peter's is checked for all pieces except one of the contributions to the C_A^2 coefficient, where an error in Peter's evaluation is pointed out. In the present paper we give the results of the calculation using the Schröder results. Note that the variation is *very* small; the reason is that the two loop static potential is only one of the several contributions to the $O(\alpha_s^2)$ calculation. For example, Using the value of the two loop coefficient found by Peter, we found

$$m_b = 5\,015^{+110}_{-70}\ \text{MeV};\ m_c = 1\,884^{+222}_{-133}\ \text{MeV}\quad (P) \tag{1.1a}$$

to which correspond the $\overline{\text{MS}}$ masses,

$$\bar m_b(\bar m_b^2) = 4\,453^{+50}_{-32}\ \text{MeV};\ \bar m_c(\bar m_c^2) = 1\,547^{+169}_{-102}\ \text{MeV}\,.\quad (P)$$

With Schröder's result, one now has

$$\begin{aligned} m_b &= 5\,001^{+104}_{-66}\ \text{MeV};\\ \bar m_b(\bar m_b^2) &= 4\,440^{+43}_{-28}\ \text{MeV}\quad (S)\end{aligned} \tag{1.1b}$$

and, for the c quark,

$$\begin{aligned} m_c &= 1\,866^{+190}_{-154}\ \text{MeV};\\ \bar m_c(\bar m_c^2) &= 1\,531^{+132}_{-127}\ \text{MeV}\quad (S)\end{aligned} \tag{1.1c}$$

For the leptonic decay of the Υ, and the hyperfine splitting the results with Schröder,s value of b_1 are

$$\Gamma(\Upsilon\to e^+e^-) = 1.07\pm 0.28\ \text{keV}\ (\text{exp.} = 1.320\pm 0.04\ \text{keV})$$

and

$$M(\Upsilon) - M(\eta) = 46.6^{+14.8}_{-12.7}\ \text{MeV}$$

they are almost identical to those obtained before.

2 The Effective Potential

We follow the method of effective potentials, and the renormalization scheme of ref. 3. The Hamiltonian for quarkonium is

$$H = H^{(0)} + H_1 \tag{2.1a}$$

where

$$\begin{aligned} H^{(0)} =& 2m + \frac{-1}{m}\Delta - \frac{C_F\tilde\alpha_s(\mu^2)}{r},\\ \tilde\alpha_s(\mu^2) =& \alpha_s(\mu^2)\left\{1 + \left(a_1 + \frac{\gamma_{\rm E}\beta_0}{2}\right)\frac{\alpha_s(\mu^2)}{\pi}\right.\\ &\left. + \left[\gamma_{\rm E}\left(a_1\beta_0 + \frac{\beta_1}{8}\right) + \left(\frac{\pi^2}{12} + \gamma_{\rm E}^2\right)\frac{\beta_0^2}{4} + b_1\right]\frac{\alpha_s^2}{\pi^2}\right\}\end{aligned} \tag{2.1b}$$

and will be solved exactly. H_1 is

$$H_1 = V_{\rm tree} + V_1^{(L)} + V_2^{(L)} + V^{(LL)} + V_{\rm s.rel} + V_{\rm spin}, \tag{2.1c}$$

and (we repeat the formulas of PYI for ease of reference)

$$\begin{aligned}
V_{\rm tree} =& \frac{-1}{4m^3}\Delta^2 + \frac{C_F\alpha_s}{m^2 r}\Delta, \\
V_1^{(L)} =& \frac{-C_F\beta_0\alpha_s(\mu^2)^2}{2\pi}\,\frac{\log r\mu}{r}, \\
V_2^{(L)} =& \frac{-C_F\alpha_s^3}{\pi^2}\left(a_1\beta_0 + \frac{\beta_1}{8} + \frac{\gamma_{\rm E}\beta_0^2}{2}\right)\frac{\log r\mu}{r} \\
\equiv& \frac{-C_F c_2^{(L)}\alpha_s^3}{\pi^2}\,\frac{\log r\mu}{r}, \\
V^{(LL)} =& \frac{-C_F\beta_0^2\alpha_s^3}{4\pi^2}\,\frac{\log^2 r\mu}{r}, \\
V_{\rm s.rel} =& \frac{C_F a_2\alpha_s^2}{2mr^2}, \\
V_{\rm spin} =& \frac{4\pi C_F\alpha_s}{3m^2}s(s+1)\delta(\mathbf{r}).
\end{aligned} \tag{2.1d}$$

Here the running coupling constant has to be taken to three loops. a_1 was calculated in ref. 1 and a_2 in ref. 3. Both have been checked by independent calculations; the only discrepancy lies in the value of the coefficient [1] b_1. According to Peter,[5]

$$\begin{aligned}
b_1 =& \tfrac{1}{16}\Big\{ \left[\tfrac{4343}{162} + 6\pi^2 - \tfrac{1}{4}\pi^4 + \tfrac{22}{3}\zeta(3)\right] C_A^2 \\
& - \left[\tfrac{1798}{81} + \tfrac{56}{3}\zeta(3)\right] C_A T_F n_f - \left[\tfrac{55}{3} - 16\zeta(3)\right] C_F T_F n_f + \tfrac{400}{81}T_F^2 n_f^2\Big\} \\
\simeq& 24.30,
\end{aligned} \tag{2.2a}$$

while Schröder[6] gives

$$\begin{aligned}
b_1 =& \tfrac{1}{16}\Big\{ \left[\tfrac{4343}{162} + 4\pi^2 - \tfrac{1}{4}\pi^4 + \tfrac{22}{3}\zeta(3)\right] C_A^2 \\
& - \left[\tfrac{1798}{81} + \tfrac{56}{3}\zeta(3)\right] C_A T_F n_f - \left[\tfrac{55}{3} - 16\zeta(3)\right] C_F T_F n_f + \tfrac{400}{81}T_F^2 n_f^2\Big\} \\
\simeq& 13.2.
\end{aligned} \tag{2.2b}$$

Although the difference between the two lies only in one piece of the coefficient of $\frac{1}{16}C_F^2$, a $4\pi^2$ vs. a $6\pi^2$, the Schröder result makes the two loop correction much smaller (and hence the overall calculation more believable). We present here results with (2.2b) since they are the ones more likely to be correct (moreover, results with (2.2a) can be found in PYI).

It should be noted that H_1 contains a tree level velocity correction, and a velocity-dependent one loop piece, $V_{\rm s.rel}$. This is because the average velocity in a Coulombic potential is $\langle|v|\rangle \sim \alpha_s$, hence a calculation correct to order α_s^4 requires tree level $O(v^2)$ and one loop $O(|v|)$ contributions. All these terms in H_1 may be treated as perturbations to first order, *except* $V_1^{(L)}$. For this, the second order perturbative contribution is also required as this also produces a correction of order α_s^4. It is to be noted that *all* the dependence on b_1 is contained in the b_1 dependence of $\widetilde{\alpha}_s$, Eq. (2.1b).

A last comment concerns the renormalization scheme. We have followed ref. 3 in renormalizing α_s in the $\overline{\rm MS}$ scheme; but the mass m that appears in Eqs. (2.1) is the two loop pole mass. That is to say, it is defined by the equation,

$$S_2^{-1}(\not{p} = m, m) = 0 \tag{2.3}$$

where $S_2(\not{p}, m)$ is the quark propagator to two loops. One can relate m to the $\overline{\rm MS}$ parameter, also to two loop accuracy, using the results of refs. 7.

[1] For the values of the constants other than b_1 entering above formulas, cf. PYI.

Nonperturbative corrections are not included in Eqs.(2.1); they will be incorporated later.

3 Energy Shifts, Order α_s^4, Λ^4/m^4

Taking into account the expression for the Hamiltonian, Eq. (2.1), we write

$$E_{nl} = 2m - m\frac{C_F^2\tilde\alpha_s^2}{4n^2} + \sum_V \delta_V^{(1)} E_{nl} + \delta^{(2)}_{V_1^{(L)}} E_{nl} + \delta_{\rm NP} E_{nl}. \tag{3.1}$$

We define generally the analogue of the Bohr radius,

$$a(\mu^2) = \frac{2}{mC_F\tilde\alpha_s(\mu^2)},$$

and then (PYI),

$$\delta^{(1)}_{V_{\rm tree}} E_{nl} = -\frac{2}{n^3\, m^3\, a^4}\left[\frac{1}{2l+1} - \frac{3}{8n}\right] + \frac{C_F\alpha_s}{m^2}\,\frac{2l+1-4n}{n^4(2l+1)a^3}; \tag{3.2a}$$

$$\delta^{(1)}_{V_1^{(L)}} E_{nl} = -\frac{\beta_0 C_F\alpha_s^2(\mu^2)}{2\pi n^2 a}\left[\log\frac{na\mu}{2} + \psi(n+l+1)\right]; \tag{3.2b}$$

$$\delta^{(1)}_{V_2^{(L)}} E_{nl} = -\frac{C_F c_2^{(L)}\alpha_s^3}{\pi^2 n^2 a}\,\left[\log\frac{na\mu}{2} + \psi(n+l+1)\right]; \tag{3.2c}$$

$$\begin{aligned}\delta^{(1)}_{V^{(LL)}} E_{nl} = -\frac{C_F\beta_0^2\alpha_s^3}{4\pi^2 n^2 a}\Big\{&\log^2\frac{na\mu}{2} + 2\psi(n+l+1)\log\frac{na\mu}{2}\\ &+\psi(n+l+1)^2 + \psi'(n+l+1)\\ &+\theta(n-l-2)\frac{2\Gamma(n-l)}{\Gamma(n+l+1)}\sum_{j=0}^{n-l-2}\frac{\Gamma(2l+2+j)}{j!(n-l-j-1)^2}\Big\};\end{aligned} \tag{3.2d}$$

$$\delta^{(1)}_{V_{\rm s.rel}} E_{nl} = \frac{C_F a_2\alpha_s^2}{m}\,\frac{1}{n^3(2l+1)a^2}. \tag{3.2e}$$

For the vector states (Υ, Υ', Υ''; J/ψ, ψ', ...) one has to add the hyperfine shift, at tree level,

$$\delta^{(1)}_{V_{\rm spin}} E_{nl} = \delta_{s1}\delta_{l0}\frac{8C_F\alpha_s}{3n^3m^2a^3}. \tag{3.2f}$$

The calculation of the second order contribution of $V_1^{(L)}$, $\delta^{(2)}_{V_1^{(L)}} E_{nl}$, is nontrivial, and may be found in PYI. We write

$$\delta^{(2)}_{V_1^{(L)}} E_{nl} \equiv -m\frac{C_F^2\beta_0^2\alpha_s^4}{4n^2\pi^2}\left\{N_0^{(n,l)} + N_1^{(n,l)}\log\frac{na\mu}{2} + \tfrac{1}{4}\log^2\frac{na\mu}{2}\right\} \tag{3.3a}$$

and the N are given in PYI. These equations are unchanged from PYI; we repeat them here for ease of reference.

The dominant nonperturbative corrections are associated with the gluon condensate and are[4]

$$\begin{aligned}\delta_{\rm NP} E_{nl} = m\epsilon_{nl} n^2\pi\langle\alpha_s G^2\rangle\left(\frac{na}{2}\right)^4 = m\frac{\epsilon_{nl} n^6\pi\langle\alpha_s G^2\rangle}{(mC_F\tilde\alpha_s)^4};\\ \epsilon_{10} = \tfrac{1\,872}{1\,275},\ \epsilon_{20} = \tfrac{2\,102}{1\,326},\ \epsilon_{21} = \tfrac{9\,929}{9\,945}.\end{aligned} \tag{3.4}$$

Because $\langle\alpha_s G^2\rangle \sim \Lambda^4$, this is of order $(\Lambda/m)^4$. Besides the corrections reported above, there are a few pieces of the higher order perturbative and nonperturbative corrections that are known; they can be found discussed in PYI.

4 Numerical Results

Using the formulas deduced above one evaluates the quark mass, and spectrum and other properties of heavy quarkonium systems. We take,

$$\Lambda(n_f = 4, \text{ three loops}) = 0.23^{+0.08}_{-0.05} \text{ GeV } \left[\alpha_s(M_Z^2) \simeq 0.114^{+0.006}_{-0.004} \right], \tag{4.1a}$$

and for the gluon condensate, very poorly known,

$$\langle \alpha_s G^2 \rangle = 0.06 \pm 0.02 \text{ GeV}^4 . \tag{4.1b}$$

This value of $\alpha_s(M_Z^2)$ is slightly smaller than, though compatible with, the world average $\alpha_s(M_Z^2) = 0.118$. We have preferred our value, which is obtained averaging measurements performed at *spacelike* momenta; see the recent review of S. Bethke[8].

Another matter to be discussed is the choice of the renormalization point, μ. As our equations (3.2, 3) show, a *natural* value for this parameter is

$$\mu = \frac{2}{na}, \tag{4.2}$$

for states with the principal quantum number n, and this will be our choice. For states with $n = 1$ the results of the calculation will turn out to depend little on the value of μ, provided it is reasonably close to (4.2). Higher states are another matter; we will discuss our choices when we consider them.

The 10 state of $\bar{b}b$ and the mass m_b. As stated, we select, for the Υ state, $\mu = 2/a$. We then use Eqs. (3.1-4) to obtain the values of the b quark mass. The results are reported below; the errors correspond to the errors in Eqs. (4.1a, b). In the estimate of the errors, the condition $\mu = 2/a$ is maintained satisfied when varying Λ while for the error due to the variation of μ the other parameters are kept fixed (i.e., one no more has then $\mu = 2/a$). The dependence of m_b on μ should be taken as an indication of the theoretical uncertainty of our calculation. With Schröder's value for b_1,

$$\begin{aligned} m_b &= 5.001^{+0.097}_{-0.061}(\Lambda) \mp 0.005 (\langle \alpha_s G^2 \rangle) \, ^{-0.025}_{+0.037} \text{ (vary } \mu^2 \text{ by } 25\%) \pm 0.006 \text{ (other th. uncertainty)} \\ \bar{m}_b(\bar{m}_b^2) &= 4.440^{+0.025}_{-0.013}(\Lambda) \mp 0.005 (\langle \alpha_s G^2 \rangle) \, ^{-0.023}_{+0.035} \text{ (vary } \mu^2 \text{ by } 25\%) \pm 0.005 \text{ (other th. uncertainty).} \end{aligned} \tag{4.3}$$

The values of μ^2, $\alpha_s(\mu^2)$, $\tilde{\alpha}_s(\mu^2)$ are, respectively,

$$\mu^2 = 6.632 \text{ GeV}^2, \ \alpha_s(\mu^2) = 0.246 \, , \tilde{\alpha}_s(\mu^2) = 0.386.$$

The piece denoted by the expression "other th. uncertainty" in (4.3) refers to the error coming from higher dimensional operators and higher order perturbative terms; it can be found discussed in PYI. It is comfortably smaller that the errors due to the uncertainty on Λ, $\langle \alpha_s G^2 \rangle$. If we omit these errors, so as not to double count them, and consider that the theoretical error is only that due to varying μ^2 by 25%, and compose all the errors quadratically, then we obtain the estimates reported in the Introduction, Eqs. (1.1).

$M(\Upsilon) - M(\eta_b)$; **the decay $\Upsilon \to e^+e^-$.** The expressions for the hyperfine splitting, and the decay of the Υ into e^+e^- are as in PYI. They depend on b_1 only indirectly, through the preferrred values of m, μ. We have the numerical results, using the Schröder calculation

$$M(\Upsilon) - M(\eta) = 46.6^{+10.9}_{-3.5}(\Lambda) \, ^{+5.5}_{-5.2} (\langle \alpha_s G^2 \rangle) \, ^{+8.3}_{-11.1} (\mu^2 = 6.632 \pm 25\%) \tag{4.4}$$

and

$$\Gamma(\Upsilon \to e^+e^-) = 1.07^{+0.11}_{+0.01}(\Lambda) \, ^{+0.12}_{-0.11} (\langle \alpha_s G^2 \rangle) \, ^{+0.21}_{-0.26} (\mu^2 = 6.632 \pm 25\%). \tag{4.5}$$

practically unchanged from PYI. Note that, when varying Λ, $\langle \alpha_s G^2 \rangle$, we have varied m_b according to Eq. (4.3), but we have *not* varied m_b when varying μ. Note also that the corrections are here fairly large; in particular, due to the large size of the radiative correction to the decay.[9]

Higher order NP corrections due to the higher dimensional operators are also known for the decay rate (see PYI). Size corrections, however, are not known now.

The result for the decay is in reasonable agreement with experiment,

$$\Gamma_{\rm exp.}(\Upsilon \to e^+e^-) = 1.320 \pm 0.04 \text{ keV}\,.$$

Higher states ($n = 2$) of $\bar{b}b$. As is clear from the expressions (3.2, 3) the natural choice of scale is now $\mu = 1/a = 2.860$ GeV^2. If we take this, adding or subtracting a 25% to estimate the dependence of the calculation on the choice of scale then we obtain the results, with the Schröder value of b_1 are,

$$\begin{aligned} M(20,\text{ th}) - M(20,\text{ exp}) &= 363^{+310}_{-324}\text{ MeV } (\mu^2 = 2.86 \pm 25\%),\\ M(21,\text{ th}) - M(21,\text{ exp}) &= 208^{+205}_{-216}\text{ MeV } (\mu^2 = 2.86 \pm 25\%). \end{aligned} \tag{4.6}$$

The 10 state of $\bar{c}c$ and the mass m_c. The value of the parameter Λ used now, corresponding to that in Eq. (4.1a), is

$$\Lambda(n_f = 3, \text{ three loops}) = 0.30^{+0.09}_{-0.05}\text{ GeV}\,.$$

The values for the c quark mass, deduced from the J/ψ mass are now,

$$\begin{aligned} m_c &= 1.866^{+0.154}_{-0.091}(\Lambda) \mp 0.014\,(\langle\alpha_s G^2\rangle)\,{}^{-0.124}_{+0.110}\text{ (varying } \mu^2 \text{ by } 25\%) \pm 0.011 \text{ (th. uncertainty)}\\ \bar{m}_c(\bar{m}_c^2) &= 1.531^{+0.083}_{-0.052}(\Lambda) \mp 0.013\,(\langle\alpha_s G^2\rangle)\,{}^{-0.115}_{+0.102}\text{ (varying } \mu^2 \text{ by } 25\%) \pm 0.010 \text{ (th. uncertainty)} \quad (S), \end{aligned} \tag{4.7}$$

and $\mu^2 = 2.465$ GeV^2 now. Composing the errors as for the b-quark we find the results reported in the Introduction, Eqs. (1.1).

5 Discussion

The discussion of PYI holds valid for the calculations using both Peter's and Schröder's evaluations, with one point of difference. If we believe Schröder's value of b_1, then the two loop corrections are comfortably small. For example, with Peter's value we had

$$a_1\alpha_s/\pi \simeq 0.11, \quad b_1\alpha_s^2/\pi^2 \simeq 0.14, \quad (P)$$

while with Schröder's the first is almost unchanged but the second becomes $\quad b_1\alpha_s^2/\pi^2 \simeq 0.081 \quad (S)$

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